

# PERSISTENT QUERIES

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**ABSTRACT.** We propose a syntax and semantics for interactive abstract state machines to deal with the following situation. A query is issued during a certain step, but the step ends before any reply is received. Later, a reply arrives, and later yet the algorithm makes use of this reply. By a persistent query, we mean a query for which a late reply might be used. Syntactically, our proposal involves issuing, along with a persistent query, a location where a late reply is to be stored. Semantically, it involves only a minor modification of the existing theory of interactive small-step abstract state machines.

## 1. INTRODUCTION

An abstract state machine (ASM) describes an algorithm by telling what it does in any one step. A run of an ASM is the result of repeatedly executing the one-step instructions, possibly interleaved with interventions from the environment. See [4] for details or see Section 4 below for a summary.

Previous theoretical work on ASMs has concentrated on what happens during a single step. For example, the papers [5, 1, 2, 3] established, for various classes of algorithms, the theorem that every algorithm in the class can be matched, step for step, by an ASM. Almost nothing was said there about what happens between steps, because almost nothing can be said; the environment can make essentially arbitrary inter-step changes to the state.

Intra-step interaction with the environment, in contrast, was treated in great detail in [2, 3]. The key difference from inter-step interaction is that, although the environment can, during a step, give essentially arbitrary replies to the algorithm's queries, the effect of these replies on the state and thus on the future course of the computation is under the algorithm's control.

In the present paper, we use inter-step interaction to treat an issue arising out of intra-step interaction, namely the possibility of a query

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being answered after the completion of the step in which the query was issued. We describe an extension of ASM syntax to accommodate such late replies, and we relate it to the ASMs of [3].

As in [5, 2, 3], we restrict attention to small-step — also known as sequential — algorithms. The amount of work that a small-step algorithm performs during any one step is bounded independently of the state or input. In the rest of the article, algorithms are by default small-step. In [2, Part I], we argued that, in principle, the intra-step interaction of an algorithm with the environment reduces to the algorithm querying the environment and the environment answering these queries. In the case of ordinary algorithms [2], every query issued during a step needs to be answered before the algorithm finishes the step. In the general case [3], however, the algorithm may finish a step without having all the replies.

If, in such a situation, the reply to a query arrives after the algorithm's step has ended, then the question arises how to handle the late reply. It may happen that the algorithm does not need that late reply; consider for example an algorithm that issues two queries and sets  $x$  to 1 when at least one of the two replies arrives. In such a case, the late reply can simply be ignored or discarded. But suppose that the algorithm eventually, at some later step, needs the late reply. In the framework of [3] where attention is restricted to one step of an algorithm, the natural solution was this. If and when the algorithm needs a late reply, it issues an auxiliary query inquiring whether the reply is in. Another possibility, close to current programming practice, is to fork out a separate computation thread that will wait for the late reply and will perhaps do some work with the late reply if and when it appears [6]. Here we propose a new solution that does not require additional queries or additional computation threads.

We base our discussion on the model of interactive computation introduced and analyzed in [3], which we review in Section 4. This model differs from the earlier, more special model of [2] in two ways, one of which is the possibility of completing a computation step without waiting for replies to all the queries issued during the step.<sup>1</sup> It is this possibility that opens the door to the topic of late replies and their subsequent use by the algorithm. The primary purpose of this paper is to describe an ASM model that incorporates such *persistent queries* and their *late replies*.

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<sup>1</sup>The other is that the algorithm can take into account the order in which replies are received.

The paper is organized as follows. We begin in Section 2 with some examples showing the relevance of late replies. In Section 3, we briefly describe our proposed extension of the traditional ASM syntax to handle persistent queries and late replies. This description is intended to convey the general idea, without presupposing details about the traditional syntax and semantics. Those details and the associated semantical notions are reviewed in Section 4, in preparation for a more careful presentation of our proposal. Section 5 discusses in more detail the possibility of finishing a step while some of that step’s queries remain unanswered. Finally, Section 6 presents in detail our ASM model for this situation, and shows how these ASMs can be represented in the model from [3].

As indicated above, we limit ourselves here to small-step algorithms, i.e., algorithms that work in discrete steps (as opposed to distributed algorithms where there may be no clear notion of (global) step because agents act asynchronously) and do only a bounded amount of work per step, with the bound depending only on the algorithm, not on the input or state (as opposed to, for example, massively parallel algorithms where the number of available processors may be increased according to the input size).

There are several justifications for this limitation. First, many of the algorithms used in practice are small-step.

Second, even in massively parallel or distributed algorithms, the individual processors or agents are usually small-step algorithms. Communication between agents is, from the point of view of any one agent, an interaction with its environment. An important motivation for developing a general model for interaction between an algorithm and its environment is this situation where the algorithm under consideration is one agent — a small-step algorithm — while the environment includes the other agents.

Third, small-step algorithms are the only class of algorithms for which the general theory of interactive algorithms has been worked out in detail and for which interactive ASMs have been proved to be adequate to capture all algorithms of the class [2, 3]. For parallel algorithms, the analogous work has been done only in the absence of intra-step interaction [1], and the case of distributed algorithms remains entirely in the domain of future work. Thus, the foundation on which we shall build in the present paper is currently available only for small-step algorithms.

Finally, it is reasonable to expect that what we do here for small-step algorithms will suggest how to do the analogous tasks for broader classes of algorithms, once the necessary framework is in place. What

we do here may also be useful in extending our work on parallel algorithms [1] to include external interactions, because it allows greater flexibility in handling the flood (or trickle) of replies that a parallel algorithm might receive all at once.

## PART I: AN IMPROVED INTERACTIVE SMALL-STEP ASM MODEL

In this part we explain a new model of interactive small-step abstract state machines that allows us to handle persistent queries. The explanation covers the syntax and its intended meaning. In the second part of the paper we cover the formal semantics of the new model in full detail.

### 2. PERSISTENT QUERIES AND LATE REPLIES

As indicated earlier, we are concerned in this paper with providing an ASM formalism that conveniently handles the following situation: An algorithm issues a query during a certain step, but finishes the step without getting an answer to that query. The answer arrives later and is then used in some subsequent step of the algorithm.

By a *late reply*, we mean a reply from the environment to a query  $q$ , reaching the algorithm after the completion of the step in which  $q$  was issued. If a late reply to  $q$  can influence the subsequent work of the algorithm, then we call  $q$  a *persistent query*.

*Remark 1.* The most natural meaning of “can influence” in the preceding sentence involves what can actually happen in runs of the algorithm. Like other run-time properties, persistence is then undecidable in general. That undecidability does no harm to our work in this paper. On the other hand, when writing programs, one is faced with the need to decide which queries should be considered persistent and treated by the methods of this paper. For this purpose, one should interpret “can influence” to mean that the programmer does not know with certainty that a late answer will never be used. It does no harm if a program treats a query as persistent even when, at run time, it turns out not to be persistent.

*Example 2.* We revisit Example 3.3 of [3, Part I]. In that example, a broker has a block of shares to sell and offers the entire block to two clients. As soon as he gets a positive reply from either client, he sells all the shares to that client. (The situation where positive replies from both clients reach the broker simultaneously is discussed in [3, Part I, Example 3.20], but it need not concern us here.) Suppose that the

broker has sold the shares to client A and completed his step (with an update to his state, recording the sale) without having received any reply from client B. Later, he gets a reply from B, who also wants to buy the shares. He should then tell B, “Sorry, I already sold the shares to someone else, whose acceptance of my offer reached me before yours.” Thus, the actions of the broker (regarded as an algorithm) take into account B’s reply, even though the reply came after the completion of the step in which the associated query (the offer to sell the shares) was issued. So that query is persistent.

*Example 3.* A pollster sends questionnaires to many people. Being a small-step algorithm, the pollster sends the questionnaires a few at a time, so the sending occupies numerous steps. Later, the filled-in questionnaires arrive and the pollster processes them. Usually, a questionnaire will be filled in and returned only after the end of the (pollster’s) step in which it was sent out. So the filled-in questionnaires are late replies, and the associated queries, the original, blank questionnaires, are persistent queries.

Even if one of the respondents is so quick that the pollster gets the reply in the same step in which he issued the query (so we are dealing with a traditional reply, as in [2, 3], not a late reply), the pollster will probably want to postpone processing this reply until after he finishes mailing all the questionnaires. More generally, an algorithm may well treat all replies the same, whether they are late or not.

How should persistent queries and late replies be treated in the context of ASMs?

Recall (from [4] or [2] or [3] — see Section 4 below for a review) how queries arise and how their answers are used in the computation done by an ASM. Queries are produced by terms  $f(t_1, \dots, t_n)$  in the ASM program, where  $f$  is an external function symbol. The queries and replies from any single step of the computation form a history (in [3], an answer function in [2]), which is empty at the beginning of a step and gradually grows as queries are issued and answered. The history influences the algorithm’s actions (issuing additional queries, ending the step, updating the state) during that step, but it is reset to empty for the start of the next step. This is in accordance with the general principle — intuitively the definition of “state” — that the state must include all the information from past steps that can influence the future progress of the computation.

As in previous work [2, 3], we adopt the so-called Lipari convention, namely that if the same external function symbol occurs several times

in an ASM program, and if its arguments at different occurrences happen to evaluate to the same elements in a particular state, then all those occurrences result in only a single query during a single step of the algorithm. For a discussion of alternative conventions and of our reasons for adopting the Lipari convention, see [2, Part II, Section 4].

*Remark 4.* One of those alternative conventions, the must-vary convention, is commonly used in practice. This convention requires all occurrences of external functions in a program to produce different queries, even if they involve the same function with the same arguments. The idea is that, whenever a query is issued, an additional component, an ID, is added automatically, and all these IDs are distinct. Thus, even if two queries look the same to the ASM, the IDs make them distinct. (In [2], the must-vary convention applied to the queries issued within a single step, but we naturally take it to also apply to queries from different steps. All the IDs are distinct, whether from the same step or not.) It appears that, given a suitable formalization of ASM semantics under the must-vary convention, what we do in this paper would work under that convention as well. We do not attempt to develop the must-vary version of the theory here, but we shall add occasional remarks about how this convention would affect our discussion.

With this rough description of the situation (see Section 4 for a detailed description), we turn to the question of handling persistent queries and late replies in the context of ASMs.

Perhaps the first approach that comes to mind is that, when an algorithm wants to use a late reply to a previously issued query  $q$ , it simply re-issues  $q$ . Then the late reply to the old  $q$  would appear, in the history of the later step, as the reply to the new  $q$ .

*Remark 5.* Under the must-vary convention, this approach would not arise, since there would be no such thing as re-issuing a query.

The trouble with this approach is that re-issuing an old query already has a different meaning: It is an entirely new query, not related (in general) to the previous query. In particular, if  $q$  is issued and answered at some step and then issued again at a later step, it may get an entirely different reply the second time. In other words, the histories that occur in different steps of a computation are not required to agree in any way. Recall that each step of an algorithm's computation begins with an empty history and gradually builds up to a larger history as queries are issued and answered, but at the end of the step, its history disappears, so there is no connection between the query  $q$  issued at two different steps. Formally, this fact is incorporated in the the definition

of coherence and the Step Postulate in [3] (reviewed in Section 4 below), which make no allowance for any influence of the histories of earlier steps. Informally, the same fact is a consequence of the general principle that all the information from the computation's past that can affect its future must be in the state, not in some other memory of histories from previous steps.

An algorithm, having issued  $q$  in some earlier step but having received no answer, might well want to both use a late reply to that  $q$  and also issue  $q$  anew for a possibly different reply. Obviously, this situation cannot be modeled by using a re-issued  $q$  to represent looking for a late reply.

The same difficulty can also be seen in the two examples above. If the broker looked for a late reply from client B by re-issuing the query, then this would look to B like a new offer to sell a (possibly different) block of shares. Similarly, for a pollster to look at the replies he has received is quite different from sending out the questionnaires again.

A second approach also uses queries whereby the algorithm looks for late replies, but these queries will not be repetitions of the original queries. Instead, this approach is similar to the use of implicit queries, which was introduced in [2, Part I, Section 2] as a way to represent an algorithm's paying attention to unsolicited information from the environment. (See also [2, Part II, Example 5.14] and [3, Part I, Remark 3.7].) The idea here is that, when it wants to use a late reply to a query  $q$ , the algorithm is, in effect, asking the environment to provide that late reply, if one exists. That is, the algorithm issues a query asking, "What late reply, if any, has been received for the query  $q$ ?"

*Remark 6.* Unlike the first approach, the second makes sense in the must-vary context. The following paragraph would, however, be modified. Instead of needing tags in case the same query is issued at several previous steps (an impossibility under must-vary), the algorithm would need to know the IDs that were attached to its previous queries.

Actually, this query needs to be more detailed. The query  $q$  could have been issued at several earlier steps, and these occurrences of  $q$  would be treated by the environment as distinct queries, which could receive different answers. So late replies might be available for several of these occurrences. The algorithm needs to say which occurrence it wants. So the implicit query might have the form "What late reply, if any, has been received for the query  $q$  that I issued in step  $n$ ?" To avoid the need for both the algorithm and the environment to count steps, the algorithm might assign some tags to persistent queries at the time it issues them, and inform the environment about the tags, so



that it can later ask “What late reply, if any, has been received for the query  $q$  that I issued with tag  $t$ ?”

A version of this approach was suggested in [2, Part I, Section 2], even though the algorithms of that paper never finished a step with unanswered queries. Nevertheless, the possibility of an query-reply pair spanning several steps was addressed as follows. The query should be regarded as a simple message to the environment, whose answer received in the same step is an uninformative “OK,” and the “real” answer in some later step should be regarded as a message from the environment, which is formally regarded as the reply to an implicit query “I’m willing to receive a message.” This version leaves it up to the environment to say which old query it is answering with its new message.

The use of new queries to request late replies to old queries has some drawbacks. It requires additional work from the environment, namely storing all late replies until the algorithm asks for them, and then delivering them immediately. This produces a mismatch between the ASM model and what would ordinarily happen in practice. A real environment would probably deliver a late reply as soon as it is available and expect the algorithm to deal with it from then on.

If a reply is not yet available when requested, the algorithm might well keep issuing the same request, step after step, and just reading all those requests would be a burden for the environment. This hardly matters as long as we take the algorithm’s point of view and regard the environment as given. But it would matter in a distributed algorithm, where an agent’s environment consists of other agents and the algorithms executed by those agents would have to include ways of handling a barrage of queries for which the answer isn’t available.

Such approaches also clash, on a more philosophical level, with the standard ASM notion of state. Once a late reply is available, it is something that resulted from the past steps of the computation and may be relevant to the future; so it ought to be part of the state.

A third approach, quite common in practice, is the use of futures, new threads created to allow the algorithm (or the parent thread) to proceed without waiting for late replies to persistent queries. A future could receive a late reply and either report it to the parent thread (or some other thread) or do some other work with it. It could also do additional work both before and after the late reply arrives. This approach would take us out of the realm of small-step algorithms for two reasons. First, the new threads need not be synchronized with the parent thread or with each other, so we would no longer have a global state advancing step by step. Second, even if we demanded



synchronization, a situation could arise where an algorithm has issued a great many queries at various steps in the past (only a few queries at any one step, if the algorithm is small-step) and has a great many futures waiting for the replies. If these futures do any computing while they wait, the total work they do might not be bounded. Since we want to remain in the framework of small-step algorithms, we do not adopt futures as our method of handling late replies.

We therefore prefer a fourth approach, in which a late reply is recorded directly in the algorithm's state. The next section explores this approach in somewhat more detail.

### 3. ASMS WITH PERSISTENT QUERIES

In this section, we discuss the last approach mentioned above for handling persistent queries and late replies. When a late reply becomes available, the environment should record it in the state of the algorithm.

The environment's action of recording the late reply, since it takes place without a new query from the algorithm, is an inter-step interaction. It directly updates the algorithm's state, without any action by the algorithm. So this update cannot occur earlier than first inter-step moment after the reply becomes available. It might occur later, if the environment is busy with other tasks or if communication is slow. Fortunately, this makes no difference to our discussion (though it may make a difference to the efficiency of the algorithm). Indeed, from the point of view of an algorithm (or ASM) it makes no difference if a late reply, received at the start of a certain step, was actually available much earlier to the environment; the algorithm simply doesn't see such availability. As far as the algorithm is concerned, the only notion of "available" is "delivered to me by the environment."

If this method of communication between the environment and the algorithm is to succeed, they must agree as to where, in the algorithm's state, a late reply to a particular query is to be recorded. The environment must know where to put the reply, and the algorithm must know where to find the reply when needed. We propose that this agreement be achieved as follows. Whenever it issues a query for which a late reply might be relevant (a persistent query), the algorithm should give the environment, along with the query, a *reply location*, where any late reply to this query should be recorded.

As in [4] (and all subsequent work on ASMs), we take *location* to mean a pair  $\langle f, \mathbf{a} \rangle$  where  $f$  is a function symbol of the algorithm's vocabulary and  $\mathbf{a}$  is a tuple of elements of the state, an  $n$ -tuple if  $f$  is  $n$ -ary.

Recall (from [2, 3] or see Section 4 below) that a query is a tuple of elements of the disjoint union  $X \sqcup \Lambda$ , where  $X$  is (the underlying set of) the state and  $\Lambda$  is a set of labels. If the function symbol  $f$  is among the labels, then a location  $\langle f, \mathbf{a} \rangle$  is almost a query. “Almost” because the location is  $\langle f, \langle a_1, \dots, a_n \rangle \rangle$  while the query is  $\langle f, a_1, \dots, a_n \rangle$ ; we shall ignore such bracketing distinctions in the future and write as if locations are queries.

It is not enough, however, for the algorithm to issue, along with any persistent query, its reply location as a second query (an **Output** in the sense of [2] or an **issue** in the sense of [3], to which the environment gives an automatic, immediate, and uninformative reply). The algorithm must tell the environment which reply location goes with which query. After all, the algorithm might issue many queries simultaneously.

The simplest way for the algorithm to convey the necessary information to the environment is to issue, along with any persistent query  $q$ , a second query that contains both  $q$  and the reply location. We adopt, by convention, the following format for this second query. It is the concatenation of three sequences:

- the query  $q$ ,
- the one-term sequence  $\langle \mathbf{rl} \rangle$ , and
- the reply location  $l$ .

Here the special label **rl** (abbreviating “reply location”) marks where the query ends and the reply location begins (and indicates that there *is* a reply location, i.e., that this is not just another query); we assume that this marker **rl** is chosen to be distinct from all other labels used by the algorithm.

Here a simplification is possible, if the environment is willing to cooperate. The information in the original query  $q$  is repeated in the first part of the additional message  $\langle q, \mathbf{rl}, l \rangle$  that specifies the reply location  $l$ . So there is no real need to issue  $q$ ; it would suffice to issue  $\langle q, \mathbf{rl}, l \rangle$  if the environment is smart enough to interpret it as follows: Regard the part before **rl** as a query in the traditional sense, but, if the reply is late, then put it into the location given after **rl**.

As a further simplification, we adopt the convention that the reply to a persistent query should be put into its reply location even if the reply arrives during the step in which the query was issued. We impose no requirement, however, on how soon such a reply is put into the reply location. It need not happen at the end of the step in which the query was issued; it might happen at the end of some later step. The reason for this flexibility is that we do not wish to impose requirements on how

fast the environment works, or even on the relative speed of different parts of the environment. Thus, one part of the environment may be able to provide an immediate reply directly to the algorithm (as in [2]) while the part of the environment responsible for inter-step changes to the algorithm's state is slower. Fortunately, this flexibility does no harm to our theory.

Instead of having the environment put on-time replies into the reply location, we could program algorithms so that, when a persistent query is issued and answered during the same step, the algorithm writes the reply into the reply location. Our convention relieves the algorithm of this duty, assigning it to the environment instead.

Does this reassignment unduly burden the environment? One can argue that it actually makes the environment's job easier. If only late replies are to be written to the reply location, then the environment must watch the step-by-step progress of the algorithm's work, in order to know whether a particular reply is late. With our convention, the environment need not monitor the algorithm in such detail; all replies to persistent queries go into the assigned reply locations. The only difference between on-time and late replies is that the former are seen by the algorithm, in its history (or answer function), without having to wait until the end of the step.

We propose the following ASM syntax for generating the combined queries — persistent query combined with reply location. (The same syntax could also be used in a framework where  $q$  and  $\langle q, \mathbf{rl}, l \rangle$  are issued separately.) Suppose the query  $q$  results from a term  $g(\mathbf{u})$  in an ASM. So  $g$  is an  $m$ -ary external function symbol for some  $m$  and  $\mathbf{u}$  is an  $m$ -tuple of terms  $u_i$ ;  $q$  results from inserting the values (in the algorithm's current state) of the  $u_i$ 's in the template associated to  $g$ . (Recall from [2, Part II, Section 4.2] that a template is like a query but with placeholders instead of elements of the state. An ASM provides, for each external function symbol  $g$ , a template  $\hat{g}$ . The query issued by  $g$  with arguments  $u_i$  is obtained by replacing the placeholders in  $\hat{g}$  by the values of the  $u_i$ 's. See also Section 4 below.) Suppose further that the desired location for late replies is  $\langle f, \mathbf{a} \rangle$ . The components  $a_j$  of the tuple  $\mathbf{a}$  must be the values, in the current state, of some terms  $t_j$ , in order for the algorithm to be able to refer to them. Then the algorithm can specify the desired location by means of the term  $f(\mathbf{t})$ . To say, in an ASM program, that the algorithm should ask the query arising from  $g(\mathbf{u})$  and to specify  $\langle f, \mathbf{a} \rangle$  as its reply location, we write

$$g(\mathbf{u})[=: f(\mathbf{t})].$$

For human readability, the brackets indicate that the main query here is produced by  $g(\mathbf{u})$ , and the reverse-assignment notation  $=:$  indicates that  $f(\mathbf{t})$  is to be read as specifying a location (like the left side of an update rule written with  $:=$ ) and that the value to be put there is the (eventual) value of  $g(\mathbf{u})$ .

In the situation described here, since  $\langle f, \mathbf{a} \rangle$  is a location, the function symbol  $f$  must be in the state vocabulary, not an external function symbol. In fact, we require all function symbols in the terms  $\mathbf{t}$  to be from the state vocabulary also. That is, the ASM should not need to issue queries in order to determine reply locations. This requirement arises from the combination of two circumstances. First, a reply location for a query should be determined when the query is issued, not in some later step. (It would be a serious problem if the query location were determined only after the arrival of the reply that should go into this location.) So any queries arising from external function symbols in  $\mathbf{t}$  need to be answered in the current step, not later. Second, it turns out that whether a query must be answered in the current step depends only on the context in which it appears in the ASM program. (The relevant contexts are timing guards, guards built with Kleene connectives, and issue rules. In all other contexts, queries must be answered in the current step. This will be proved formally in Proposition 47 below.) But a persistent query  $g(\mathbf{u})$  and its reply location  $f(\mathbf{t})$  share the same context. So if the latter must be answered in the current step, so must the former. And then the former doesn't need a reply location.

*Remark 7.* We could relax this requirement and allow  $\mathbf{t}$  to issue queries provided we have some assurance, from a source other than the context in the ASM program, that these queries will be answered in the current step. Such assurance could come from knowledge about the environment. It could also come from other parts of the ASM program. A simple example of the latter possibility is given by the program

```

if  $t = t$  then
  if  $c \prec g(u)[=: f(t)]$  then
     $x := 0$ 
  endif
endif.
```

Here  $t$ ,  $c$ , and  $g$  are external but  $f$  is in the state vocabulary. A step of this algorithm can finish without a value for  $g(u)$ , provided  $c$  has a value. But it cannot finish without a value for  $t$ , because of the guard  $t = t$  (whose sole purpose is to require that  $t$  have a value).

*Example 8.* Consider a simplified version of the broker example as detailed in [3, Part I, Example 3.20]; the purpose of the simplification is to avoid hiding the currently relevant topic, persistent queries and late replies, in a sea of other considerations. We regard the broker's offers to the two clients (whom we name 0 and 1) as given by nullary external functions  $q_0$  and  $q_1$  (instead of ternary functions having the stock, the number of shares, and the price as arguments), and we assume the broker breaks ties (when he gets positive answers from both clients simultaneously) in favor of client 0 (rather than non-deterministically or randomly). Also, we assume that, as long as the broker has received no answer from either client, or has a negative answer from one client and no answer from the other, he simply waits. The resulting algorithm is represented by the following ASM, in the notation of [3, Part II]. We assume that the dynamic function symbols  $s_0$  and  $s_1$  are used to indicate a sale to client 0 or 1, respectively, so they have the value **false** in initial states.

```

if  $\neg$  Halt then
  do in parallel
    if  $s_0 = s_1 = \text{false} \wedge q_0 = \text{true} \wedge (q_0 \preceq q_1 \vee q_1 = \text{false})$ 
      then  $s_0 := \text{true}$  endif
    if  $s_0 = s_1 = \text{false} \wedge q_1 = \text{true} \wedge (q_1 \prec q_0 \vee q_0 = \text{false})$ 
      then  $s_1 := \text{true}$  endif
    if  $s_0 = s_1 = \text{false} \wedge q_0 = \text{false} \wedge q_1 = \text{false}$ 
      then skip endif
    Halt := true
  enddo
endif

```

The so-called Kleene conjunction  $\wedge$  and Kleene disjunction  $\vee$  that are used in this ASM program are like ordinary conjunction  $\wedge$  and disjunction  $\vee$  except that  $p \wedge q$  is false as soon as one conjunct is false, even if the other is undefined, and dually for  $\vee$ . For more details, see [3, Part II, Section 2.3] or Section 4 below.

**Convention 9.** In future examples, we shall omit “if  $\neg$ Halt then” and the associated “endif”, adopting instead the convention that an ASM program is to be executed repeatedly until **Halt** becomes true. This convention supersedes the one from [4] that the iteration continues until there is no change of state from one step to the next. The new convention allows an algorithm to continue waiting for a late reply without making any changes to its state.

Now suppose, as in Example 2 above, we want the algorithm to respond to a late reply from a losing client with a letter explaining that the shares have already been sold. We assume (again for simplicity, to avoid hiding the relevant issues) that the broker's vocabulary contains nullary symbols  $l_0$  and  $l_1$  denoting appropriate letters to the two clients. And we assume that it also has nullary symbols  $a_0$  and  $a_1$ , initially denoting `undef`, to be used as the reply locations. Then the modified algorithm, which behaves like the one above but also sends the appropriate letter, is given in our proposed syntax by the following ASM.

```

do in parallel
  if  $s_0 = s_1 = \text{false} \wedge q_0 = \text{true} \wedge (q_0 \preceq q_1[=: a_1] \vee q_1 = \text{false})$ 
    then  $s_0 := \text{true}$  endif
  if  $s_0 = s_1 = \text{false} \wedge q_1 = \text{true} \wedge (q_1 \prec q_0[=: a_0] \vee q_0 = \text{false})$ 
    then  $s_1 := \text{true}$  endif
  if  $s_0 = s_1 = \text{false} \wedge q_0 = \text{false} \wedge q_1 = \text{false}$ 
    then skip endif
  if  $s_0 = \text{true} \wedge a_1 = \text{true}$  then issue( $l_1$ ) endif
  if  $s_1 = \text{true} \wedge a_0 = \text{true}$  then issue( $l_0$ ) endif
  if  $(a_0 = \text{true} \vee a_0 = \text{false}) \wedge (a_1 = \text{true} \vee a_1 = \text{false})$ 
    then Halt := true endif
enddo

```

The first two lines have been modified by attaching reply locations  $a_i$  to the two query-producing terms  $q_i$ . (It doesn't really matter which occurrence of  $q_i$  is annotated with  $a_i$ . We chose to use the occurrence that is primarily responsible for the possibility of finishing the step without a reply.) Two new lines have been added, containing instructions for issuing the appropriate letter to the losing client. The last line makes the algorithm end its run when both clients have answered; until then, even if the shares have been sold to one client, it waits for an answer from the other client.

This example serves to illustrate a general feature of our notation. The part of the program that tells what to do with late replies to the queries  $q_i$  does not mention those queries at all. Rather, it mentions the locations  $a_i$  where the late replies are to be found. The executor of the algorithm need not remember, when using a late reply, the query that it answers; only the location of the late reply is relevant, and it is used like any other location in the state.

*Remark 10.* The example also has a somewhat special property, namely that it doesn't need modes. It is common, in ASM programs, to use

certain nullary, dynamic symbols as modes, to keep track of what sort of work the algorithm is currently doing. In the present example, there would be two modes, one indicating that the broker is waiting for a positive reply in order to sell the stock, and one indicating that the stock has been sold to one of the clients but the broker may still need to send a letter to the other client. It is often convenient to include such modes and update them explicitly in an ASM program. In the present case, however, this would be redundant, as the first mode is already described by  $s_0 = s_1 = \mathbf{false}$  and the second mode by the negation of this.

*Example 11.* Consider the pollster example, 3. Let us assume that the pollster sends out  $N$  questionnaires, numbered from 0 to  $N-1$ , that the replies will be numbers, and that the desired output is the sum of all these numbers. For simplicity, we also assume that the questionnaires are sent one at a time and that all the replies eventually arrive, though perhaps late and out of order; our pollster algorithm will keep running without producing an output until all the replies have been received and added. The pollster first sends out all the questionnaires (using an internal variable  $i$  to keep track of where he is in this process) and then goes through all the replies, adding them one at a time (re-using  $i$  to keep track of this process as well). We describe what the pollster does as an ASM, using the following vocabulary. As already indicated,  $i$  is a dynamic, nullary symbol ranging from 0 to  $N-1$  and indexing the queries and their replies; it is initially 0. An additional dynamic, nullary symbol **all-sent**, initially **false**, tells whether all the questionnaires have been sent. Unary functions  $q$  and  $l$  send each  $i$  to the  $i^{\text{th}}$  questionnaire  $q(i)$  and its reply location  $l(i)$ . The initial value of  $l(i)$  is **undef** for each  $i$ . A dynamic, nullary function **sum**, initially 0, represents, at each step, the sum of the replies that have been added so far. Elementary arithmetic is assumed to be available, particularly  $+$ ,  $<$ , and names for specific numbers. Here is the ASM:



```

do in parallel
  if all-sent = false  $\wedge i < N$  then do in parallel
    issue( $q(i)[=: l(i)]$ )
     $i := i + 1$ 
  enddo endif
  if all-sent = false  $\wedge i = N$  then do in parallel
     $i := 0$ 
    all-sent := true
  enddo endif
  if all-sent = true  $\wedge i < N \wedge l(i) \neq \text{undef}$  then do in parallel
    sum := sum +  $l(i)$ 
     $i := i + 1$ 
  enddo endif
  if all-sent = true  $\wedge i = N$  then Halt := true endif
enddo

```

Recall here Convention 9 that a run of the ASM ends when `Halt` becomes true; until then the program is executed repeatedly. We also assume that `Halt` is initially false, so that the program runs.

*Remark 12.* When justifying the “query and reply” paradigm for intra-step interaction in [2, Part I, Section 2], we wrote that, if an algorithm sends a message to the outside world without expecting a reply, then this situation can be modeled by imagining an automatic, immediate, and uninformative reply “OK,” essentially just an acknowledgment that the message was sent. The `Output` rules in [2, Part II] and the `issue` rules in [3, Part II] were introduced to produce such messages. There is, however, nothing in the official semantics in [2] or [3] to require the environment to produce only “OK” as a reply to such queries. Although an `issue` rule cannot make use of any nontrivial information provided by its reply, nothing prohibits the existence of such information.

In fact, there are situations where such nontrivial information is to be expected, for example in

```

do in parallel
   $x := q$ 
  issue( $q$ )
enddo

```

(where  $q$  is an external nullary symbol and  $x$  an internal dynamic one). In this (admittedly silly) program, the query produced by the `issue`

line is also produced, with the intention of using its reply, by the update rule  $x := q$ .

Following the official semantics given for ASMs in [3, Part II], we make no special assumptions about the replies to queries that result from **issue** rules. These replies can be any elements of the state, just as for any other queries.

This convention was used in Example 11, because the queries produced by **issue**( $q(i)[=: l(i)]$ ) are the questionnaires, whose replies should be the numbers stored in locations  $l(i)$  and then added.

This example also used the earlier convention, whereby replies go into the reply locations even if they are not late. Without this convention, the ASM program would have to include instructions whereby, if an answer to  $q(i)$  appears in the same step in which the query was issued, the algorithm would put that answer into location  $l(i)$ .

*Remark 13.* Futures can provide a particular way of implementing our approach to late replies. A future that simply waits for a late reply and, when one arrives, writes it into the appropriate reply location thereby accomplishes what we require of the environment. Nevertheless, there is a conceptual difference. By assigning to the environment the task of putting the late replies into the proper locations, we maintain sequentiality of the algorithm. By assigning the same task to futures, a part of the algorithm, one enters the more complex domain of asynchronous, distributed algorithms.

## PART II: THE DETAILS

### 4. INTERACTIVE ABSTRACT STATE MACHINES

We now begin a more formal treatment of ASMs with persistent queries. We build on the ASM model described in [3]. In the present section, we summarize the material from [3] that we need here. This summary also serves to explain things that were taken for granted in the preceding sections. We do not, however, repeat the extensive discussion offered in [3] to motivate and explain the model.

We begin by recalling the definitions, conventions, and postulates for interactive small-step algorithms. This material is taken from [3, Part I, Section 3].

**States Postulate:** The algorithm determines

- a finite vocabulary  $\Upsilon$ ,
- a nonempty set  $\mathcal{S}$  of *states*, which are  $\Upsilon$ -structures,
- a nonempty subset  $\mathcal{I} \subseteq \mathcal{S}$  of *initial states*,

- a finite set  $\Lambda$  of *labels* (to be used in forming queries).

As in earlier papers, we use the following conventions concerning vocabularies and structures.

**Convention 14.**

- A vocabulary  $\Upsilon$  consists of function symbols with specified arities.
- Some of the symbols in  $\Upsilon$  may be marked as *static*, and some may be marked as *relational*. Symbols not marked as static are called *dynamic*.
- Among the symbols in  $\Upsilon$  are the logic names: nullary symbols **true**, **false**, and **undef**; unary **Boole**; binary equality; and the usual propositional connectives. All of these are static and all but **undef** are relational.
- An  $\Upsilon$ -structure  $X$  consists of a nonempty base set, usually denoted by the same symbol  $X$ , and interpretations of all the function symbols  $f$  of  $\Upsilon$  as functions  $f_X$  on that base set.
- In any  $\Upsilon$ -structure, the interpretations of **true**, **false**, and **undef** are distinct.
- In any  $\Upsilon$ -structure  $X$ , the interpretations of relational symbols are functions whose values lie in  $\{\mathbf{true}_X, \mathbf{false}_X\}$ .
- In any  $\Upsilon$ -structure  $X$ , the interpretation of **Boole** maps **true** <sub>$X$</sub>  and **false** <sub>$X$</sub>  to **true** <sub>$X$</sub>  and everything else to **false** <sub>$X$</sub> .
- In any  $\Upsilon$ -structure  $X$ , the interpretation of equality maps pairs of equal elements to **true** <sub>$X$</sub>  and all other pairs to **false** <sub>$X$</sub> .
- In any  $\Upsilon$ -structure  $X$ , the propositional connectives are interpreted in the usual way when their arguments are in  $\{\mathbf{true}_X, \mathbf{false}_X\}$ , and they take the value **false** <sub>$X$</sub>  whenever any argument is not in  $\{\mathbf{true}_X, \mathbf{false}_X\}$ .
- We may omit subscripts  $X$ , for example from **true** and **false**, when there is no danger of confusion.  $\square$

**Definition 15.** A *potential query* in state  $X$  is a finite tuple of elements of  $X \sqcup \Lambda$ . A *potential reply* in  $X$  is an element of  $X$ .  $\square$

Here  $X \sqcup \Lambda$  means the disjoint union of  $X$  and  $\Lambda$ . So if they are not disjoint, then they are to be replaced by disjoint isomorphic copies. We shall usually not mention these isomorphisms; that is, we write as though  $X$  and  $\Lambda$  were disjoint.

**Definition 16.** An *answer function* for a state  $X$  is a partial map from potential queries to potential replies. A *history* for  $X$  is a pair  $\xi = \langle \dot{\xi}, \leq_\xi \rangle$  consisting of an answer function  $\dot{\xi}$  together with a linear

pre-order  $\leq_\xi$  of its domain. By the *domain* of a history  $\xi$ , we mean the domain  $\text{Dom}(\dot{\xi})$  of its answer function component, which is also the field of its pre-order component.  $\square$

Recall that a *pre-order* of a set  $D$  is a reflexive, transitive, binary relation on  $D$ , and that it is said to be *linear* if, for all  $x, y \in D$ ,  $x \leq y$  or  $y \leq x$ . The equivalence relation defined by a pre-order is given by

$$x \equiv y \iff x \leq y \leq x.$$

The equivalence classes are partially ordered by

$$[x] \leq [y] \iff x \leq y,$$

and this partial order is linear if and only if the pre-order was.

We also write  $x < y$  to mean  $x \leq y$  and  $y \not\leq x$ . (Because a pre-order need not be antisymmetric,  $x < y$  is in general a stronger statement than the conjunction of  $x \leq y$  and  $x \neq y$ .) When, as in the definition above, a pre-order is written as  $\leq_\xi$ , we write the corresponding equivalence relation and strict order as  $\equiv_\xi$  and  $<_\xi$ . The same applies to other subscripts and superscripts.

We use histories to express the information received by the algorithm from its environment during a step. The answer function part  $\dot{\xi}$  of a history tells what replies the environment has given to the algorithm's queries, and the pre-order part  $\leq_\xi$  tells in what order these replies were received. Specifically, if  $q$  is in the domain of  $\xi$ , then  $\dot{\xi}(q)$  is the environment's answer to the query  $q$ . If  $p, q \in \text{Dom}(\xi)$  and  $p <_\xi q$ , this means that the answer  $\dot{\xi}(p)$  to  $p$  was received strictly before the answer  $\dot{\xi}(q)$  to  $q$ . If  $p \equiv_\xi q$ , this means that the two answers were received simultaneously.

We emphasize that the timing we are concerned with here is logical time, not physical time. That is, it is measured by the progress of the computation, not by an external clock. In particular, we regard a query as being issued by the algorithm as soon as the information causing that query (in the sense of the Interaction Postulate below) is available. This is why we include, in histories, only the relative ordering of replies. The ordering of queries relative to replies or relative to each other is then determined. The logical time of a query is the same as the logical time of the last of the replies needed to cause that query.

**Definition 17.** Let  $\leq$  be a pre-order of a set  $D$ . An *initial segment* of  $D$  with respect to  $\leq$  is a subset  $S$  of  $D$  such that whenever  $x \leq y$  and  $y \in S$  then  $x \in S$ . An *initial segment* of  $\leq$  is the restriction of  $\leq$  to an initial segment of  $D$  with respect to  $\leq$ . An *initial segment* of a history  $\langle \dot{\xi}, \leq_\xi \rangle$  is a history  $\langle \dot{\xi} \upharpoonright S, \leq_\xi \upharpoonright S \rangle$ , where  $S$  is an initial segment

of  $\text{Dom}(\dot{\xi})$  with respect to  $\leq_\xi$ . (We use the standard notation  $\upharpoonright$  for the restriction of a function or a relation to a set.) We write  $\eta \trianglelefteq \xi$  to mean that the history  $\eta$  is an initial segment of the history  $\xi$ . If  $q \in D$ , then we define two associated initial segments as follows.

$$\begin{aligned} (\leq q) &= \{d \in D : d \leq q\} \\ (< q) &= \{d \in D : d < q\}. \quad \square \end{aligned}$$

**Interaction Postulate** For each state  $X$ , the algorithm determines a binary relation  $\vdash_X$ , called the *causality relation*, between finite histories and potential queries.

The intended meaning of  $\xi \vdash_X q$  is that, if the algorithm's current state is  $X$  and the history of its interaction so far (as seen by the algorithm during the current step) is  $\xi$ , then it will issue the query  $q$  unless it has already done so in the current step. When we say that the history so far is  $\xi$ , we mean not only that the environment has given the replies indicated in  $\dot{\xi}$  in the order given by  $\leq_\xi$ , but also that no other queries have been answered. Thus, although  $\xi$  explicitly contains only positive information about the replies received so far, it also implicitly contains the negative information that there have been no other replies. Of course, if additional replies are received later, so that the new history has  $\xi$  as a proper initial segment, then  $q$  is still among the issued queries, because it was issued at the earlier time when the history was only  $\xi$ . This observation is formalized as follows.

**Definition 18.** For any state  $X$  and history  $\xi$ , we define sets of queries

$$\begin{aligned} \text{Issued}_X(\xi) &= \{q : (\exists \eta \trianglelefteq \xi) \eta \vdash_X q\} \\ \text{Pending}_X(\xi) &= \text{Issued}_X(\xi) - \text{Dom}(\dot{\xi}). \quad \square \end{aligned}$$

Thus,  $\text{Issued}_X(\xi)$  is the set of queries that have been issued by the algorithm, in state  $X$ , by the time the history is  $\xi$ , and  $\text{Pending}_X(\xi)$  is the subset of those that have, as yet, no replies.

The following definition describes the histories that are consistent with the given causality relation. Informally, these are the histories where every query in the domain has a legitimate reason, under the causality relation, for being there.

**Definition 19.** A history  $\xi$  is *coherent*, with respect to a state  $X$  or its associated causality relation  $\vdash_X$ , if  $\text{Dom}(\dot{\xi})$  is finite and

$$(\forall q \in \text{Dom}(\dot{\xi})) q \in \text{Issued}_X(\xi \upharpoonright (< q))$$

□

*Remark 20.* In [3, Part I, Definition 3.12], the definition of coherence did not require  $\text{Dom}(\xi)$  to be finite; instead, it had the weaker requirement that the linear order of  $\equiv_\xi$ -classes induced by  $\leq_\xi$  is a well-order. The stronger requirement of finiteness was, however, deduced later from the Bounded Work Postulate for all attainable histories; see [3, Part I, Corollary 3.28]. Since we omit such deductions here, it seems clearer to build finiteness explicitly into the notion of coherent history.

**Definition 21.** A history  $\xi$  for a state  $X$  is *complete* if  $\text{Pending}_X(\xi) = \emptyset$ .  $\square$

The terminology reflects the fact that, if a complete history has arisen in the course of a computation, then there will be no further interaction with the environment during this step. No further interaction can originate with the environment, because no queries remain to be answered. No further interaction can originate with the algorithm, since  $\xi$  and its initial segments don't cause any further queries. So the algorithm must either terminate its run (successfully) if **Halt** becomes true, or proceed to the next step (by updating its state), or fail. The next definitions and postulates describe these end-of-step matters. They do not explicitly mention termination (other than by failure), but this is covered anyway, since updates are covered and termination amounts to an update of **Halt** to the value **true**.

**Definition 22.** A *location* in a state  $X$  is a pair  $\langle f, \mathbf{a} \rangle$  where  $f$  is a dynamic function symbol from  $\Upsilon$  and  $\mathbf{a}$  is a tuple of elements of  $X$ , of the right length to serve as an argument for the function  $f_X$  interpreting the symbol  $f$  in the state  $X$ . The *value* of this location in  $X$  is  $f_X(\mathbf{a})$ . An *update* for  $X$  is a pair  $(l, b)$  consisting of a location  $l$  and an element  $b$  of  $X$ . An update  $(l, b)$  is *trivial* (in  $X$ ) if  $b$  is the value of  $l$  in  $X$ . We often omit parentheses and brackets, writing locations as  $\langle f, a_1, \dots, a_n \rangle$  instead of  $\langle f, \langle a_1, \dots, a_n \rangle \rangle$  and writing updates as  $\langle f, \mathbf{a}, b \rangle$  or  $\langle f, a_1, \dots, a_n, b \rangle$  instead of  $(\langle f, \mathbf{a} \rangle, b)$  or  $(\langle f, \langle a_1, \dots, a_n \rangle \rangle, b)$ .  $\square$

The intended meaning of an update  $\langle f, \mathbf{a}, b \rangle$  is that the interpretation of  $f$  is to be changed (if necessary, i.e., if the update is not trivial) so that its value at  $\mathbf{a}$  is  $b$ .

**Step Postulate — Part A** The algorithm determines, for each state  $X$ , a set  $\mathcal{F}_X$  of *final histories*. Every complete, coherent history has an initial segment (possibly the whole history) in  $\mathcal{F}_X$ .

Intuitively, a history is final for  $X$  if, whenever it arises in the course of a computation in  $X$ , the algorithm completes its step, either by

failing or by executing its updates and proceeding to the next step or terminating the run if `Halt` has become true.

**Definition 23.** A history for a state  $X$  is *attainable* (in  $X$ ) if it is coherent and no proper initial segment of it is final.  $\square$

The attainable histories are those that can occur under the given causality relation and the given choice of final histories. That is, not only are the queries answered in an order consistent with  $\vdash_X$  (coherence), but the history does not continue beyond where  $\mathcal{F}_X$  says it should stop.

**Step Postulate — Part B** For each state  $X$ , the algorithm determines that certain histories *succeed* and others *fail*. Every final, attainable history either succeeds or fails but not both.

**Definition 24.** We write  $\mathcal{F}_X^+$  for the set of successful final histories and  $\mathcal{F}_X^-$  for the set of failing final histories.

The intended meaning of “succeed” and “fail” is that a successful final history is one in which the algorithm finishes its step and performs a set of updates of its state, while a failing final history is one in which the algorithm cannot continue — the step ends, but there is no next state, not even a repetition of the current state. Such a situation can arise if the algorithm computes inconsistent updates. It can also arise if the environment gives inappropriate answers to some queries.

**Step Postulate — Part C** For each attainable history  $\xi \in \mathcal{F}_X^+$  for a state  $X$ , the algorithm determines an *update set*  $\Delta^+(X, \xi)$ , whose elements are updates for  $X$ . It also produces a *next state*  $\tau(X, \xi)$ , which

- has the same base set as  $X$ ,
- has  $f_{\tau(X, \xi)}(\mathbf{a}) = b$  if  $\langle f, \mathbf{a}, b \rangle \in \Delta^+(X, \xi)$ , and
- otherwise interprets function symbols as in  $X$ .

**Convention 25.** In notations like  $\mathcal{F}_X$ ,  $\mathcal{F}_X^+$ ,  $\mathcal{F}_X^-$ ,  $\Delta^+(X, \xi)$ , and  $\tau(X, \xi)$ , we may omit  $X$  if only one  $X$  is under discussion. We may also add the algorithm  $A$  as a superscript if several algorithms are under discussion.  $\square$

Any isomorphism  $i : X \cong Y$  between states can be extended in an obvious, canonical way to act on queries, answer functions, histories, locations, updates, etc. We use the same symbol  $i$  for all these extensions.



**Isomorphism Postulate** Suppose  $X$  is a state and  $i : X \cong Y$  is an isomorphism of  $\Upsilon$ -structures. Then:

- $Y$  is a state, initial if  $X$  is.
- $i$  preserves causality, that is, if  $\xi \vdash_X q$  then  $i(\xi) \vdash_Y i(q)$ .
- $i$  preserves finality, success, and failure, that is,  $i(\mathcal{F}_X^+) = \mathcal{F}_Y^+$  and  $i(\mathcal{F}_X^-) = \mathcal{F}_Y^-$ .
- $i$  preserves updates, that is,  $i(\Delta^+(X, \xi)) = \Delta^+(Y, i(\xi))$  for all histories  $\xi$  for  $X$ .

**Convention 26.** In the last part of this postulate, and throughout this paper, we adopt the convention that an equation between possibly undefined expressions is to be understood as implying that if either side is defined then so is the other.  $\square$

### Bounded Work Postulate

- There is a bound, depending only on the algorithm, for the lengths of the tuples in  $\text{Issued}_X(\xi)$ , for all states  $X$  and final, attainable histories  $\xi$ .
- There is a bound, depending only on the algorithm, for the cardinality  $|\text{Issued}_X(\xi)|$ , for all states  $X$  and final, attainable histories  $\xi$ .
- There is a finite set  $W$  of  $\Upsilon$ -terms (possibly involving variables), depending only on the algorithm, with the following property. Suppose  $X$  and  $X'$  are two states and  $\xi$  is a history for both of them. Suppose further that each term in  $W$  has the same value in  $X$  as in  $X'$  when the variables are given the same values in  $\text{Range}(\xi)$ . Then:
  - If  $\xi \vdash_X q$  then  $\xi \vdash_{X'} q$  (so in particular  $q$  is a query for  $X'$ ).
  - If  $\xi$  is in  $\mathcal{F}_X^+$  or  $\mathcal{F}_X^-$ , then it is also in  $\mathcal{F}_{X'}^+$  or  $\mathcal{F}_{X'}^-$ , respectively.
  - $\Delta^+(X, \xi) = \Delta^+(X', \xi)$ .

**Definition 27.** An *interactive, small-step algorithm* is any entity satisfying the States, Interaction, Step, Isomorphism, and Bounded Work Postulates.  $\square$

Since these are the only algorithms under consideration in most of this paper, we often omit “interactive, small-step.”

**Definition 28.** A set  $W$  with the property required in the third part of the Bounded Work Postulate is called a *bounded exploration witness*

for the algorithm. Two pairs  $(X, \xi)$  and  $(X', \xi)$ , consisting of states  $X$  and  $X'$  and a single  $\xi$  that is a history for both, are said to *agree* on  $W$  if, as in the postulate, each term in  $W$  has the same value in  $X$  as in  $X'$  when the variables are given the same values in  $\text{Range}(\xi)$ .  $\square$

This completes our review of the notion of interactive small-step algorithm, as defined in [3, Part I]. This notion will be slightly modified in Section 6 to accommodate our proposal for handling persistent queries. A modification is needed because, when an algorithm issues a combination  $\langle q, \mathbf{rl}, l \rangle$  of a query  $q$  and a reply location  $l$ , the reply (if received in the same step) is a reply to  $q$ . So it is  $q$ , not  $\langle q, \mathbf{rl}, l \rangle$ , that should appear in the domain of the history. See Section 6 for more details.

We now turn to the notion of abstract state machine (ASM) from Part II of [3]. ASMs describe algorithms, and the main result of [3] is that all algorithms (as defined above) are behaviorally equivalent (in a very strong sense defined in [3, Part I, Section 4]) to ASMs. We begin our review of ASMs by summarizing the syntactic definitions from [3, Part II, Section 2]; afterward, we shall also summarize the semantics.

An ASM uses a vocabulary  $\Upsilon$ , subject to Convention 14, and a set  $\Lambda$  of labels as in our discussion of algorithms above. In addition, it has an *external vocabulary*  $E$ , consisting of finitely many *external function symbols*. These symbols are used syntactically exactly like static, non-relational symbols from  $\Upsilon$ , but their semantics will be quite different. If  $f$  is an  $n$ -ary external function symbol and  $\mathbf{a}$  is an  $n$ -tuple of arguments from a state  $X$ , then the value of  $f$  at  $\mathbf{a}$  is not stored as part of the structure of the state but is obtained from the environment as the reply to a query. If the history contains no reply to this query, then  $f$  has no value at  $\mathbf{a}$ .

**Definition 29.** The set of *terms* is the smallest set containing  $f(t_1, \dots, t_n)$  whenever it contains  $t_1, \dots, t_n$  and  $f$  is an  $n$ -ary function symbol from  $\Upsilon \cup E$ . (The basis of this recursive definition is, of course, given by the 0-ary function symbols.) A *Boolean term* is a term of the form  $f(\mathbf{t})$  where  $f$  is a relational symbol.  $\square$

**Convention 30.** By  $\Upsilon$ -*terms*, we mean terms built using the function symbols in  $\Upsilon$  and variables. These are terms in the usual sense of first-order logic for the vocabulary  $\Upsilon$ . They occur, for example, in the Bounded Work Postulate as elements of the bounded exploration witness. Terms as defined above, using function symbols from  $\Upsilon \cup E$  but not using variables, will be called *ASM-terms* when we wish to emphasize the distinction from  $\Upsilon$ -terms. A term of the form  $f(\mathbf{t})$  where  $f \in E$  is called a *query-term*.

We introduce timing explicitly into the formalism with the notation  $(s \preceq t)$ , which is intended to mean that the replies needed to evaluate the term  $s$  arrived no later than those needed to evaluate  $t$ . As explained in [3],  $\preceq$  differs from function symbols in that  $s \preceq t$  can have a truth value even when only one of  $s$  and  $t$  has a value.

We also use a version of the Boolean connectives with similar behavior, so that, for example, a disjunction counts as true as soon as one of the disjuncts is, even if the other disjunct has no truth value. This behavior characterizes the connectives of Kleene's strong three-valued logic. We use the notations  $\wedge$  and  $\vee$  for the conjunction and disjunction of this logic; the traditional conjunction and disjunction, which have values only when both constituents do, will continue to be written  $\wedge$  and  $\vee$ .

**Definition 31.** The set of *guards* is defined by the following recursion.

- Every Boolean term is a guard.
- If  $s$  and  $t$  are terms, then  $(s \preceq t)$  is a guard.
- If  $\varphi$  and  $\psi$  are guards, then so are  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$ , and  $\neg\varphi$ .

□

**Definition 32.** The set of ASM *rules* is defined by the following recursion.

- If  $f \in \Upsilon$  is a dynamic  $n$ -ary function symbol, if  $t_1, \dots, t_n$  are terms, and if  $t_0$  is a term that is Boolean if  $f$  is relational, then

$$f(t_1, \dots, t_n) := t_0$$

is a rule, called an *update* rule.

- If  $f \in E$  is an external  $n$ -ary function symbol and if  $t_1, \dots, t_n$  are terms, then

$$\text{issue } f(t_1, \dots, t_n)$$

is a rule, called an *issue* rule.

- **fail** is a rule.
- If  $\varphi$  is a guard and if  $R_0$  and  $R_1$  are rules, then

$$\text{if } \varphi \text{ then } R_0 \text{ else } R_1 \text{ endif}$$

is a rule, called a *conditional* rule.  $R_0$  and  $R_1$  are its *true* and *false* branches, respectively.

- If  $k$  is a natural number (possibly zero) and if  $R_1, \dots, R_k$  are rules then

$$\text{do in parallel } R_1, \dots, R_k \text{ enddo}$$

is a rule, called a *parallel combination* or *block* with the subrules  $R_i$  as its *components*.

□

We may omit the end-markers `endif` and `enddo` when they are not needed, for example in very short rules or in programs formatted so that indentation makes the grouping clear.

The correspondence between external function calls and queries is mediated by a template assignment, defined as follows.

**Definition 33.** For a fixed label set  $\Lambda$ , a *template* for  $n$ -ary function symbols is any tuple in which certain positions are filled with labels from  $\Lambda$  while the rest are filled with the *placeholders*  $\#1, \dots, \#n$ , occurring once each. We assume that these placeholders are distinct from all the other symbols under discussion ( $\Upsilon \cup E \cup \Lambda$ ). If  $Q$  is a template for  $n$ -ary functions, then we write  $Q[a_1, \dots, a_n]$  for the result of replacing each placeholder  $\#i$  in  $Q$  by the corresponding  $a_i$ . □

Thus if the  $a_i$  are elements of a state  $X$  then  $Q[a_1, \dots, a_n]$  is a potential query in  $X$ .

**Definition 34.** For a fixed label set and external vocabulary, a *template assignment* is a function assigning to each  $n$ -ary external function symbol  $f$  a template  $\hat{f}$  for  $n$ -ary functions. □

The intention, which will be formalized in the semantic definitions below, is that when an ASM evaluates a term  $f(t_1, \dots, t_n)$  where  $f \in E$ , it first computes the values  $a_i$  of the terms  $t_i$ , then issues the query  $\hat{f}[a_1, \dots, a_n]$ , and finally uses the answer to this query as the value of  $f(t_1, \dots, t_n)$ .

By assigning templates to external function symbols, rather than to their occurrences in a rule, we incorporate into our framework the “Lipari convention” of [2, Part II, Section 4.3]. This means that, if an external function symbol has several occurrences in an ASM program and if its arguments have the same values at these occurrences, then only a single query will be issued in any one step as a result of all of these occurrences. See Sections 4.3–4.6 of [2, Part II] for a discussion of alternative conventions, and see [2, Part III, Section 7] for additional information comparing these conventions.

**Definition 35.** An *interactive, small-step, ASM program*  $\Pi$  consists of

- a finite vocabulary  $\Upsilon$ ,
- a finite set  $\Lambda$  of labels,
- a finite external vocabulary  $E$ ,
- a rule  $R$ , using the vocabularies  $\Upsilon$  and  $E$ , the *underlying rule* of  $\Pi$ ,

- a template assignment with respect to  $E$  and  $\Lambda$ .

**Convention 36.** We use the following abbreviations:

$(s \prec t)$	for	$\neg(t \preceq s)$ ,
$(s \approx t)$	for	$(s \preceq t) \wedge (t \preceq s)$ ,
$(s \succeq t)$	for	$(t \preceq s)$ , and
$(s \succ t)$	for	$(t \prec s)$

We abbreviate the empty block `do in parallel enddo` as `skip`. We may omit parentheses when no confusion results.  $\square$

This completes the syntax of ASMs; we turn next to the semantics, as presented in [3, Part II, Section 3]. We treat terms, guards, and rules in turn. Their semantics are defined in the presence of a state  $X$ , a template assignment, and a history  $\xi$ .

The semantics of terms specifies, by induction on terms  $t$ , the queries that are caused by  $\xi$  under the associated causality relation  $\vdash_X^t$  and sometimes also a value  $\text{Val}(t, X, \xi) \in X$ . In the case of query-terms, the semantics may specify also a query called the query-value  $\text{q-Val}(t, X, \xi)$ . Evaluation of a query-term  $t$  should first issue the query  $\text{q-Val}(t, X, \xi)$ ; the reply, if any, to this query is the actual value  $\text{Val}(t, X, \xi)$  of  $t$ .

**Definition 37** (Semantics of Terms). Let  $t$  be the term  $f(t_1, \dots, t_n)$ .

- (1) If  $\text{Val}(t_i, X, \xi)$  is undefined for at least one  $i$ , then  $\text{Val}(t, X, \xi)$  is also undefined, and  $\xi \vdash_X^t q$  if and only if  $\xi \vdash_X^{t_i} q$  for at least one  $i$ . If  $f \in E$  then  $\text{q-Val}(t, X, \xi)$  is also undefined.
- (2) If, for each  $i$ ,  $\text{Val}(t_i, X, \xi) = a_i$  and if  $f \in \Upsilon$ , then  $\text{Val}(t, X, \xi) = f_X(a_1, \dots, a_n)$ , and no query  $q$  is caused by  $\xi$ .
- (3) If, for each  $i$ ,  $\text{Val}(t_i, X, \xi) = a_i$ , and if  $f \in E$ , then  $\text{q-Val}(t, X, \xi)$  is the query  $\hat{f}[a_1, \dots, a_n]$ .
  - If  $\text{q-Val}(t, X, \xi) = q \in \text{Dom}(\dot{\xi})$ , then  $\text{Val}(t, X, \xi) = \dot{\xi}(q)$ , and no query is caused by  $\xi$ .
  - If  $\text{q-Val}(t, X, \xi) = q \notin \text{Dom}(\dot{\xi})$ , then  $\text{Val}(t, X, \xi)$  is undefined, and  $q$  is the unique query such that  $\xi \vdash_X^t q$ .

$\square$

The semantics of guards, unlike that of terms, depends not only on the answer function but also on the preorder in the history. Another difference from the term case is that the values of guards, when defined, are always Boolean values.

**Definition 38** (Semantics of guards). Let  $\varphi$  be a guard and  $\xi$  a history in an  $\Upsilon$ -structure  $X$ .

- (1) If  $\varphi$  is a Boolean term, then its value (if any) and causality relation are already given by Definition 37.
- (2) If  $\varphi$  is  $(s \preceq t)$  and if both  $s$  and  $t$  have values with respect to  $\xi$ , then  $\text{Val}(\varphi, X, \xi) = \mathbf{true}$  if, for every initial segment  $\eta \preceq \xi$  such that  $\text{Val}(t, X, \eta)$  is defined,  $\text{Val}(s, X, \eta)$  is also defined. Otherwise,  $\text{Val}(\varphi, X, \xi) = \mathbf{false}$ . Also declare that  $\xi \vdash_X^\varphi q$  for no  $q$ .
- (3) If  $\varphi$  is  $(s \preceq t)$  and if  $s$  has a value with respect to  $\xi$  but  $t$  does not, then define  $\text{Val}(\varphi, X, \xi)$  to be  $\mathbf{true}$ ; again declare that  $\xi \vdash_X^\varphi q$  for no  $q$ .
- (4) If  $\varphi$  is  $(s \preceq t)$  and if  $t$  has a value with respect to  $\xi$  but  $s$  does not, then define  $\text{Val}(\varphi, X, \xi)$  to be  $\mathbf{false}$ ; again declare that  $\xi \vdash_X^\varphi q$  for no  $q$ .
- (5) If  $\varphi$  is  $(s \preceq t)$  and if neither  $s$  nor  $t$  has a value with respect to  $\xi$ , then  $\text{Val}(\varphi, X, \xi)$  is undefined, and  $\xi \vdash_X^\varphi q$  if and only if  $\xi \vdash_X^s q$  or  $\xi \vdash_X^t q$ .
- (6) If  $\varphi$  is  $\psi_0 \wedge \psi_1$  and both  $\psi_i$  have value  $\mathbf{true}$ , then  $\text{Val}(\varphi, X, \xi) = \mathbf{true}$  and no query is produced.
- (7) If  $\varphi$  is  $\psi_0 \wedge \psi_1$  and at least one  $\psi_i$  has value  $\mathbf{false}$ , then  $\text{Val}(\varphi, X, \xi) = \mathbf{false}$  and no query is produced.
- (8) If  $\varphi$  is  $\psi_0 \wedge \psi_1$  and one  $\psi_i$  has value  $\mathbf{true}$  while the other,  $\psi_{1-i}$ , has no value, then  $\text{Val}(\varphi, X, \xi)$  is undefined, and  $\xi \vdash_X^\varphi q$  if and only if  $\xi \vdash_X^{\psi_{1-i}} q$ .
- (9) If  $\varphi$  is  $\psi_0 \wedge \psi_1$  and neither  $\psi_i$  has a value, then  $\text{Val}(\varphi, X, \xi)$  is undefined, and  $\xi \vdash_X^\varphi q$  if and only if  $\xi \vdash_X^{\psi_i} q$  for some  $i$ .
- (10) The preceding four clauses apply with  $\vee$  in place of  $\wedge$  and  $\mathbf{true}$  and  $\mathbf{false}$  interchanged.
- (11) If  $\varphi$  is  $\neg\psi$  and  $\psi$  has a value, then  $\text{Val}(\varphi, X, \xi) = \neg\text{Val}(\psi, X, \xi)$  and no query is produced.
- (12) If  $\varphi$  is  $\neg\psi$  and  $\psi$  has no value then  $\text{Val}(\varphi, X, \xi)$  is undefined and  $\xi \vdash_X^\varphi q$  if and only if  $\xi \vdash_X^\psi q$ .

□

The semantics of a rule, for an  $\Upsilon$ -structure  $X$ , an appropriate template assignment, and a history  $\xi$ , consists of a *causality relation*, declarations of whether  $\xi$  is *final* and whether it *succeeds* or *fails*, and a set of *updates*.

**Definition 39** (Semantics of Rules). Let  $R$  be a rule and  $\xi$  a history for the  $\Upsilon$ -structure  $X$ . In the following clauses, whenever we say that a history succeeds or that it fails, we implicitly also declare it to be final;

contrapositively, when we say that a history is not final, we implicitly also assert that it neither succeeds nor fails.

- (1) If  $R$  is an update rule  $f(t_1, \dots, t_n) := t_0$  and if all the  $t_i$  have values  $\text{Val}(t_i, X, \xi) = a_i$ , then  $\xi$  succeeds for  $R$ , and it produces the update set  $\{\langle f, \langle a_1, \dots, a_n \rangle, a_0 \rangle\}$  and no queries.
- (2) If  $R$  is an update rule  $f(t_1, \dots, t_n) := t_0$  and if some  $t_i$  has no value, then  $\xi$  is not final for  $R$ , it produces the empty update set, and  $\xi \vdash_X^R q$  if and only if  $\xi \vdash_X^{t_i} q$  for some  $i$ .
- (3) If  $R$  is **issue**  $f(t_1, \dots, t_n)$  and if all the  $t_i$  have values  $\text{Val}(t_i, X, \xi) = a_i$ , then  $\xi$  succeeds for  $R$ , it produces the empty update set, and  $\xi \vdash_X^R q$  for the single query  $q = \hat{f}[a_1, \dots, a_n]$  provided  $q \notin \text{Dom}(\xi)$ ; if  $q \in \text{Dom}(\xi)$  then no query is produced.
- (4) If  $R$  is **issue**  $f(t_1, \dots, t_n)$  and if some  $t_i$  has no value, then  $\xi$  is not final for  $R$ , it produces the empty update set, and  $\xi \vdash_X^R q$  if and only if  $\xi \vdash_X^{t_i} q$  for some  $i$ .
- (5) If  $R$  is **fail**, then  $\xi$  fails for  $R$ ; it produces the empty update set and no queries.
- (6) If  $R$  is a conditional rule **if**  $\varphi$  **then**  $R_0$  **else**  $R_1$  **endif** and if  $\varphi$  has no value, then  $\xi$  is not final for  $R$ , and it produces the empty update set.  $\xi \vdash_X^R q$  if and only if  $\xi \vdash_X^\varphi q$ .
- (7) If  $R$  is a conditional rule **if**  $\varphi$  **then**  $R_0$  **else**  $R_1$  **endif** and if  $\varphi$  has value **true** (resp. **false**), then finality, success, failure, updates, and queries are the same for  $R$  as for  $R_0$  (resp.  $R_1$ ).
- (8) If  $R$  is a parallel combination **do in parallel**  $R_1, \dots, R_k$  **enddo** then:
  - $\xi \vdash_X^R q$  if and only if  $\xi \vdash_X^{R_i} q$  for some  $i$ .
  - The update set for  $R$  is the union of the update sets for all the components  $R_i$ . If this set contains two distinct updates at the same location, then we say that a *clash* occurs (for  $R$ ,  $X$ , and  $\xi$ ).
  - $\xi$  is final for  $R$  if and only if it is final for all the  $R_i$ .
  - $\xi$  succeeds for  $R$  if and only if it succeeds for all the  $R_i$  and no clash occurs.
  - $\xi$  fails for  $R$  if and only if it is final for  $R$  and either it fails for some  $R_i$  or a clash occurs.

□

**Definition 40.** Fix a rule  $R$  endowed with a template assignment, and let  $X$  be a  $\Upsilon$ -structure and  $\xi$  be a history for  $X$ . If  $\xi$  is successful and final for  $R$  over  $X$ , then the *successor*  $\tau(X, \xi)$  of  $X$  with respect to  $R$  and  $\xi$  is defined from the update set  $\Delta^+(X, \xi)$  as in the Step Postulate, Part C.



It is easy to check (see [3, Part II, Lemma 3.18]) that  $\tau$  is well-defined;  $\Delta^+$  will not prescribe two contradictory updates of the same location under a successful, final history.

**Definition 41.** An *interactive, small-step, ASM* consists of

- an ASM program  $\Pi$  in some vocabulary  $\Upsilon$ ,
- a nonempty set  $\mathcal{S}$  of  $\Upsilon$ -structures called *states* of the ASM, and
- a nonempty set  $\mathcal{I} \subseteq \mathcal{S}$  of *initial states*,

subject to the requirements that  $\mathcal{S}$  and  $\mathcal{I}$  are closed under isomorphism and that  $\mathcal{S}$  is closed under transitions in the following sense. If  $X \in \mathcal{S}$  and if  $\xi$  is a successful, final history for  $\Pi$  in  $X$ , then the successor  $\tau(X, \xi)$  of  $X$  with respect to  $\Pi$  and  $\xi$  is also in  $\mathcal{S}$ .  $\square$

It is shown in [3, Part II] that ASMs are algorithms, in the sense defined above by the postulates, and that, conversely, all algorithms are behaviorally equivalent to ASMs.

## 5. IMPATIENCE

We call an algorithm *patient* if it never finishes a step until the environment has answered all queries from that step. (It patiently waits for answers to all its queries.) Formally, this means that all final, attainable histories are complete. Otherwise, we call the algorithm *impatient*.

A query is said to be *blocking* if, once it is issued, the algorithm's step cannot end without a reply to this query. Thus, an algorithm is patient if and only if all its queries are blocking.

In this paper, we are concerned with non-blocking queries, and specifically with the possibility that the reply to such a query may arrive and be used by the algorithm in a later step than the one that produced the query. The present section describes where, in an ASM program, non-blocking queries can originate. Of course, we must first say precisely what it means for a query to originate in a particular part of a rule — or of a term, or of a guard.

We present the material in this section in the context of the traditional ASM syntax and semantics described above. It applies equally, however, to the modified syntax and semantics that was described in Section 3 and will be formalized in Section 6. The changes we introduce do not affect the proofs in the present section.

The discussion will be simplified by the following definitions and convention.

**Definition 42.** Let  $S$  be a term or a guard or a rule, let  $X$  be a state, and let  $\xi$  be a history for  $X$ . We define

$$\begin{aligned} \text{Issued}_X^S(\xi) &= \{q : (\exists \eta \preceq \xi) \eta \vdash_X^S q\} \\ \text{Pending}_X^S(\xi) &= \text{Issued}_X^S(\xi) - \text{Dom}(\dot{\xi}). \quad \square \end{aligned}$$

Note that, in the case of a rule, this definition agrees with Definition 18 for algorithms. We are just extending the “Issued” and “Pending” notation to apply also to terms and guards (and adding the superscript  $S$ , which was unnecessary earlier because the role of  $S$  was played there by a fixed algorithm). The next definition also extends to terms and guards terminology already available for rules.

**Definition 43.** Let  $S$  be a term or a guard, let  $X$  be a state, and let  $\xi$  be a history for  $X$ . We say that the history  $\xi$  is *final* for  $S$  in  $X$  if  $\text{Val}(S, X, \xi)$  is defined.

**Convention 44.** When we speak of syntactic parts of a term, guard, or rule, we mean occurrences of those syntactic parts. Thus, for example, “subrule” really means “occurrence of subrule.”

We now define the origins of a query caused by a term or guard or rule. The definition involves going systematically through the definitions of the semantics of term, guards, and rules (Definitions 37, 38, and 39), and checking all the clauses where a query is caused. For the reader’s convenience, we append to some clauses of the following definition some additional information, in brackets, about the circumstances in which those clauses can apply. These bracketed comments can easily be verified by inspection of Definitions 37, 38, and 39.

**Definition 45.** Let  $X$  be a state, and  $\xi$  a history for it, and  $q$  a potential query.

- If  $t$  is a term  $f(t_1, \dots, t_n)$ , if  $\text{Val}(t_i, X, \xi)$  is undefined for at least one  $i$ , and if  $\xi \vdash_X^t q$ , then the origins of  $q$  in  $t$  are the origins of  $q$  in all those  $t_i$  for which  $\xi \vdash_X^{t_i} q$ .
- If  $t$  is a term  $f(t_1, \dots, t_n)$ , if  $\text{Val}(t_i, X, \xi)$  is defined for all  $i$ , and if  $\xi \vdash_X^t q$ , then  $q$  has exactly one origin in  $t$ , namely  $t$  itself. [Here  $f$  is an external function symbol and  $q$  is the  $q$ -value of  $t$ .]
- If  $\varphi$  is  $(s \preceq t)$  and  $\xi \vdash_X^\varphi q$ , then the origins of  $q$  in  $\varphi$  are its origins in  $s$  (if any, i.e., if  $\xi \vdash_X^s q$ ) and its origins in  $t$  (if any). [According to the semantics of guards, if either  $s$  or  $t$  has a value, then  $(s \preceq t)$  issues no queries. So the present clause applies only when  $\xi$  is not final for either of these terms.]

- If  $\varphi$  is  $\psi_0 \wedge \psi_1$  or  $\psi_0 \vee \psi_1$  and  $\xi \vdash_X^\varphi q$ , then the origins of  $q$  in  $\varphi$  are its origins in  $\psi_0$  (if any) and its origins in  $\psi_1$  (if any). [At most one of  $\psi_0$  and  $\psi_1$  has a value under  $\xi$ , and if one does then that value is **true** in the case of  $\wedge$  and **false** in the case of  $\vee$ .]
- If  $\varphi$  is  $\neg\psi$  and  $\xi \vdash_X^\varphi q$ , then the origins of  $q$  in  $\varphi$  are the same as in  $\psi$ .
- If  $R$  is an update rule  $f(t_1, \dots, t_n) := t_0$  and  $\xi \vdash_X^R q$ , then the origins of  $q$  in  $R$  are the origins of  $q$  in all those  $t_i$  for which  $\xi \vdash_X^{t_i} q$ .
- If  $R$  is **issue**  $f(t_1, \dots, t_n)$ , if all the  $t_i$  have values  $\text{Val}(t_i, X, \xi) = a_i$ , and if  $\xi \vdash_X^R q$ , then  $q$  has exactly one origin in  $R$ , namely  $f(t_1, \dots, t_n)$ . [Here  $f$  is an external function symbol and  $q$  is the  $q$ -value of  $f(t_1, \dots, t_n)$ .]
- If  $R$  is **issue**  $f(t_1, \dots, t_n)$ , if some  $t_i$  has no value, and if  $\xi \vdash_X^R q$ , then the origins of  $q$  in  $R$  are the origins of  $q$  in all those  $t_i$  for which  $\xi \vdash_X^{t_i} q$ .
- If  $R$  is a conditional rule **if**  $\varphi$  **then**  $R_0$  **else**  $R_1$  **endif**, if  $\varphi$  has no value under  $\xi$ , and if  $\xi \vdash_X^R q$ , then the origins of  $q$  in  $R$  are the origins of  $q$  in  $\varphi$ .
- If  $R$  is a conditional rule **if**  $\varphi$  **then**  $R_0$  **else**  $R_1$  **endif**, if  $\varphi$  has value **true** (resp. **false**), and if  $\xi \vdash_X^R q$ , then the origins of  $q$  in  $R$  are its origins in  $R_0$  (resp.  $R_1$ ).
- If  $R$  is a parallel combination **do in parallel**  $R_1, \dots, R_k$  **enddo** and if  $\xi \vdash_X^R q$ , then the origins of  $q$  in  $R$  are its origins in all those  $R_i$  for which  $\xi \vdash_X^{R_i} q$ .

In the preceding definition,  $X$  and  $\xi$  were fixed and were therefore not mentioned in the “origin” terminology. When necessary, we make them explicit by a phrase like “origin of  $q$  in  $R$  with respect to  $X$  and  $\xi$ .”

**Lemma 46.** *Let  $S$  be a term or guard or rule, let  $X$  be a state, let  $\xi$  be a history for  $X$ , and let  $q$  be a potential query in  $X$ . Then  $\xi \vdash_X^S q$  if and only if  $q$  has at least one origin in  $S$  with respect to  $X$  and  $\xi$ . Any origin of  $q$  is a query-term  $t$ , a subterm of  $S$ , with  $q\text{-Val}(t, X, \xi) = q$ . Furthermore, this  $t$  is also the (unique) origin of  $q$  in  $t$ ; in particular,  $\xi \vdash_X^t q$ .*

*Proof.* Proceed by induction, first on terms, then on guards, and finally on rules. In every case, the proof is just a comparison of the definition of “origin” with the parts of Definitions 37, 38, and 39 that describe the causality relation.  $\square$

After these preliminaries, we can look in detail at impatience, the phenomenon of an ASM’s step ending even though some of its queries

have not been answered. In more detail, the phenomenon involves five entities:

- an ASM program  $\Pi$ ,
- a state  $X$  of  $\Pi$ ,
- a history  $\xi$  that is final for  $X$  with respect to  $\Pi$  (so the step ends),
- a query  $q \in \text{Pending}_X^\Pi(\xi)$  (so  $q$  has been issued but not answered during this step), and
- an initial segment  $\eta \preceq \xi$  such that  $\eta \vdash_X^\Pi q$ .

In connection with the last of these items,  $\eta$ , recall that for  $q$  to be issued during a step where the history is  $\xi$  it must be caused by some initial segment of  $\xi$ , though not necessarily by  $\xi$  itself.

We have simplified the notation by using the same symbol  $\Pi$  for an ASM program and for its underlying rule, even though the program also includes additional materials, particularly the template assignment. This additional material will remain fixed, so our abuse of notation will not cause confusion.

The following proposition describes the possible origins of queries that remain unanswered at the end of a step. Notice that, when the  $S$  in the proposition is a rule  $\Pi$ , then the hypotheses of the proposition describe the five items listed above.

**Proposition 47.** *Let  $S$  be a term or guard or rule. Let  $X$  be a state and let  $\eta \preceq \xi$  be two histories for  $X$ , such that  $\xi$  is final for  $X$  with respect to  $S$ . Let  $q$  be a query such that  $\eta \vdash_X^S q$  but  $q \notin \text{Dom}(\dot{\xi})$ . Then all origins of  $q$  in  $S$  with respect to  $X$  and  $\eta$  are of one of the following sorts:*

- query-subterms of  $s$  or  $t$  in a timing guard  $s \preceq t$  within  $S$ ,
- query-subterms of  $\psi_0$  or  $\psi_1$  in a Kleene-conjunction  $\psi_0 \wedge \psi_1$  or Kleene-disjunction  $\psi_0 \vee \psi_1$  within  $S$ ,
- arguments  $t$  of issue-rules  $\text{issue}(t)$  within  $S$ .

*Proof.* Assume that  $S, X, \eta, \xi$ , and  $q$  are as in the hypothesis of the proposition and that  $o$  is an origin of  $q$  in  $S$  with respect to  $X$  and  $\eta$ . Assume also, as an induction hypothesis, that the proposition becomes true if  $S$  is replaced by any proper subterm, subguard, or subrule (while  $X, \eta, \xi, q$ , and  $o$  are unchanged.)

To save a little writing later, observe that the hypothesis that  $\eta \vdash_X^S q$  is redundant, because, according to Lemma 46, if it didn't hold then there would be no origin of  $q$  in  $S$  with respect to  $X$  and  $\eta$ , and so the conclusion of the proposition would hold vacuously.

The proof will repeatedly use the observation that, if the conclusion of the proposition holds when  $S$  is replaced by some subterm, subguard, or subrule  $S'$  of  $S$ , then it also holds for  $S$  itself. The reason is that the conclusion refers to  $S$  only in the context of saying that some guard or rule occurs within  $S$ . If we find the desired guard or rule within  $S'$  then we certainly have it within  $S$ .

The fact that  $\eta \vdash_X^S q$  must arise from one of the clauses of Definition 37, 38, or 39, with  $\eta$  in place of the  $\xi$  in the definition. And this clause cannot be one of the many clauses where the definition says that no query is caused, i.e., clause 2 and the first case in clause 3 of Definition 37; clauses 2, 3, 4, 6, 7, and 11 as well as the part of clause 10 analogous to clause 7 in Definition 38; and clauses 1 and 5 of Definition 39. We examine the remaining possibilities in turn, labeling them according to the clause in Definition 37, 38, or 39 that provides  $\eta \vdash_X^S q$ .

**37-1:**  $S$  is a term  $f(t_1, \dots, t_n)$  and, for at least one  $i$ ,  $\text{Val}(t_i, X, \eta)$  is undefined. According to Definition 45,  $o$  is an origin of  $q$  in some  $t_i$  (with respect to  $X$  and  $\eta$ ). For such an  $i$ ,  $\xi$  will be final with respect to  $t_i$ , i.e.,  $\text{Val}(t_i, X, \xi)$  will be defined, because otherwise, the same clause of Definition 37 (now applied to  $\xi$  rather than  $\eta$ ) would contradict the assumption that  $\xi$  is final for  $S$ . Thus, the hypotheses of the proposition are satisfied with  $t_i$  in place of  $S$ . By induction hypothesis, the conclusions of the proposition hold for  $t_i$ , and, as observed above, it immediately follows that they also hold for  $S$ . (Though it isn't needed for the proof, it may help the reader if we point out that this case cannot actually occur. Indeed, by what we have just proved, we would have a guard (involving  $\prec$  or  $\wedge$  or  $\vee$ ) or an issue-rule within a term, and this cannot happen in the ASM syntax.)

**37-3, second part:**  $S$  is a term  $f(t_1, \dots, t_n)$  where  $f$  is an external function symbol; each  $t_i$  has a value  $\text{Val}(t_i, X, \eta) = a_i$ ; and

$$q = \text{q-Val}(S, X, \eta) = \hat{f}[a_1, \dots, a_n].$$

(This clause in Definition 37 also says that  $q \notin \text{Dom}(\dot{\eta})$ , but this is immediate from the assumptions that  $q \notin \text{Dom}(\dot{\xi})$  and  $\eta \preceq \xi$ .) Lemma 3.6 of [3, Part II] gives us that, when we pass from  $\eta$  to its extension  $\xi$ , the values of the  $t_i$ 's do not change. So we have  $\text{Val}(t_i, X, \xi) = a_i$  and therefore (by the same clause of Definition 37)

$$\text{q-Val}(S, X, \xi) = \hat{f}[a_1, \dots, a_n] = q.$$

But then, since  $q \notin \text{Dom}(\xi)$ , the same clause tells us that  $\xi$  is not final for  $S$ . This contradicts the hypothesis of the proposition, so this case simply cannot arise.

**38-1:** Here  $S$  is guard that is a Boolean term, so this case is included in the cases already treated where  $S$  is a term.

**38-5:** Here  $S$  is a timing guard ( $s \preceq t$ ) (and neither of the terms  $s, t$  has a value with respect to  $\eta$ ). By Lemma 46,  $o$  is a subterm of this timing guard, and so we have the first of the three alternatives in the conclusion of the proposition.

**38-8 or 9:** Here  $S$  is a Kleene conjunction, and so its subterm  $o$  satisfies the second alternative in the proposition.

**38-10:** Here  $S$  is a Kleene disjunction, and so we again get the second alternative of the proposition.

**38-12:** Here  $S$  is  $\neg\psi$ . By definition, origins in  $S$  are the same as in  $\psi$ . Also, by definition, since  $\xi$  isn't final for  $S$ , it isn't final for  $\psi$ . Thus, the induction hypothesis applies and tells us that the conclusion of the proposition holds with  $\psi$  in place of  $S$ . But then it also holds for  $S$ .

**39-2:**  $S$  is an update rule  $f(t_1, \dots, t_n) := t_0$  and not all  $t_i$  have values with respect to  $\eta$ . The argument here is essentially the same as for case 37-1 above.  $o$  is an origin of  $q$  in some  $t_i$ , and  $\xi$  must be final for  $t_i$  as otherwise it would not be final for  $S$ . By induction hypothesis, the conclusion of the proposition holds with  $t_i$  in place of  $S$ , and therefore it also holds for  $S$ .

**39-3:** Here  $S$  is an issue-rule and, by definition of “origin,”  $o$  is its argument. So we have the third alternative in the proposition.

**39-4:** The argument here is again essentially the same as for cases 37-1 and 39-2; we spare the reader (and ourselves) a third occurrence of this same argument.

**39-6:** Here  $S$  is a conditional rule whose guard  $\varphi$  has no value with respect to  $\eta$ . By definition,  $o$  is an origin of  $q$  in  $\varphi$  with respect to  $\eta$ . Furthermore,  $\xi$  must be final for  $\varphi$ , because otherwise it could not be final for  $S$ . So the induction hypothesis applies and we get the conclusion of the proposition with  $\varphi$  in place of  $S$ , and therefore also for  $S$ .

**39-7:**  $S$  is a conditional rule **if**  $\varphi$  **then**  $R_0$  **else**  $R_1$  **endif** and  $\varphi$  has a value with respect to  $\eta$ . We assume  $\text{Val}(\varphi, X, \eta) = \text{true}$ ; the case of **false** is the same with  $R_0$  and  $R_1$  interchanged. By definition, the origins of  $q$  in  $S$  are the same as in  $R_0$ . Also, by Lemma 3.12 of [3, Part II],  $\text{Val}(\varphi, X, \xi) = \text{true}$ , so the finality of  $\xi$  for  $S$  implies that

$\xi$  is also final for  $R_0$ . So the induction hypothesis applies and gives us the conclusion of the proposition with  $R_0$  in place of  $S$ . As usual, the conclusion for  $S$  follows.

**39-8:**  $S$  is a parallel combination with components  $R_i$ . The definition of “origin” says that  $o$  is an origin of  $q$  in at least one of the  $R_i$ . And  $\xi$  must be final for that  $R_i$  because otherwise it could not be final for  $S$ . So the induction hypothesis gives us the conclusion of the proposition with  $R_i$  in place of  $S$ , and the conclusion for  $S$  follows.  $\square$

*Remark 48.* In view of Proposition 47, we can limit the use of the new syntax  $g(\mathbf{u})[=: f(\mathbf{t})]$  to the places described in the proposition, namely subterms of timing guards, of Kleene conjunctions, and of Kleene disjunctions, and arguments of issue-rules. External function symbols occurring anywhere else in an ASM program produce blocking queries, so there is no need to provide locations for late replies. And if a reply-location is provided for a blocking query, with the intention of having an on-time reply recorded there, then the program can easily be altered so that the ASM reads the reply in its history and writes it into the desired location.

*Example 49.* Here are some trivial examples showing that all the alternatives in the conclusion of Proposition 47 can occur (with  $S$  being a rule). Assume that the vocabulary has three external, nullary function symbols  $a, b, c$  and that the templates  $\hat{a}, \hat{b}, \hat{c}$  assigned to them are distinct.

For the first alternative in Proposition 47, consider the rule

$$\text{if } a \prec b \text{ then } x := 1 \text{ else } x := 2 \text{ endif.}$$

The empty history causes both  $\hat{a}$  and  $\hat{b}$ . Any history  $\xi$  with domain  $\{\hat{a}\}$  is final and has  $\hat{b}$  pending. This  $\hat{b}$  has exactly one origin, namely the unique  $b$  in the rule. (The updates of  $x$  could be replaced by certain other rules, for example **skip**, without affecting the idea.) The same program also serves as an example if the reply for  $b$  arrives before that for  $a$ ; then the history with only the reply for  $b$  is final,  $a$  is pending, and its only origin is in the timing guard  $a \prec b$ .

For the second alternative, consider

$$\text{if } (a = b) \wedge (a = c) \text{ then } x := 1 \text{ else } x := 2 \text{ endif.}$$

The empty history causes all three of  $\hat{a}, \hat{b}, \hat{c}$ . Any history  $\xi$  with domain  $\{\hat{a}, \hat{b}\}$  and with  $\xi(\hat{a}) \neq \xi(\hat{b})$  is final and has  $\hat{c}$  pending. The only origin of  $\hat{c}$  in this rule is the unique occurrence of  $c$ .

There is an analogous example with  $\vee$  in place of  $\wedge$ . Just use a  $\xi$  that gives the same reply to the two queries  $\hat{a}$  and  $\hat{b}$ .



Finally, for the third alternative, just use the rule `issue(a)`. The empty history causes  $\hat{a}$  and is final, with  $\hat{a}$  pending.

*Remark 50.* We take this opportunity to clarify Remark 3.17 of [3, Part II], which begins: “Issue rules are the only way an ASM can issue a query without necessarily waiting for an answer.” This appears to deny the possibility of the first two alternatives in Proposition 47. Indeed, if “waiting for an answer” means “waiting until an answer is received,” then the examples just given show that this is wrong. It becomes correct, however, if “waiting for an answer” means “waiting at least for a moment,” i.e., not finishing the step immediately. Note that, in the parts of Example 49 that don’t use `issue`, the pending query is caused not by the final history but by a proper initial segment. (In the notation of Proposition 47,  $\eta \neq \xi$ .) In this sense, the ASM does wait after issuing the query and before finishing the step.

Another description of what happens in these examples is that the unanswered query  $q$  is issued by the final history  $\xi$  (in the sense of being in `Issued( $\xi$ )`) but it is not caused by the final history ( $\xi \not\models q$ ). In the second sentence of Remark 3.17, we used the phrase “a history causes a rule to issue a query,” which is ambiguous in view of the difference between causing and issuing. It should be interpreted as causing, not merely issuing.

## 6. ANNOUNCING LOCATIONS FOR LATE REPLIES

In this section, we present the small modifications of [3] needed to accommodate the  $\langle q, \mathbf{rl}, l \rangle$  method, proposed in Section 3, for handling persistent queries in ASMs. Some of these modifications directly affect the syntax and semantics of ASMs or (in one case) even the notion of algorithm from [3]; we exhibit these with the heading “Modification.” If these are violated, then our ASM programs with persistent queries won’t make sense. Other modifications describe what we expect to see in programs and in the environment’s behavior. These concern either constraints on the environment or good programming practice; we label these “Intention.” If they are violated, ASM programs with persistent queries will still make sense, but it may not be the sense that was intended.

**Modification 1.** The set  $\Lambda$  of labels contains the “reply location marker” `rl`.

The purpose of this modification is of course to ensure availability of the queries  $\langle q, \mathbf{rl}, l \rangle$  that we want to use when issuing a persistent query  $q$  with reply location  $l$ . It may seem that we should also require that

all dynamic function symbols  $f$  should be among the labels, so that they can be used in the location part  $l$  of  $\langle q, \mathbf{r}l, l \rangle$ . This requirement would do no harm, but it may be overkill, since there may be many dynamic function symbols that will not be used for reply locations in a particular program. Accordingly, we do not impose this requirement but instead use the following definition to keep track of which function symbols are available to serve as the first component of a reply location.

**Definition 51.** A function symbol is *reply-available* if it is dynamic and is also a member of the set  $\Lambda$  of labels.

*Remark 52.* We have insisted here that the function-symbol component of a reply location be dynamic. This is not strictly necessary; one could imagine using a static function — one that the algorithm can never update — in this role, since the updates would be done by the environment, not by the algorithm. But it seems strange to allow this when the update is being done at the request of the algorithm.

Notice, for example, the following undesirable consequence of allowing reply locations that begin with a static function. Suppose the environment can provide echoes; that is, an algorithm can issue a query of the form “answer this with  $x$ ,” where  $x$  is an element of the state, and get reply  $x$ . Then by issuing the query

$$\text{answer this with } x [=: f(\mathbf{t})],$$

the algorithm can achieve (after the end of the current step) the effect of the update  $f(\mathbf{t}) := x$ . That should not be possible when  $f$  is static.

*Remark 53.* The definition of reply-available is designed to cohere with our convention in Section 3 about the format of the queries that provide reply locations. Had we chosen a different format, for example using some codes for the function symbols, then the definition should be modified accordingly.

Our next task is to understand, in a way that fits the general notions of algorithms and ASMs, the external function calls accompanied by reply locations. We can fit this syntactic construct

$$g(u_1, \dots, u_m) [=: f(t_1, \dots, t_n)]$$

into the ASM framework by treating it as a new external function symbol with all of  $u_1, \dots, u_m, t_1, \dots, t_n$  as arguments. That is, we require the availability of a new  $(m + n)$ -ary function symbol, which we denote by  $g[=: f]$ , and we treat  $g(\mathbf{u}) [=: f(\mathbf{t})]$  as syntactic sugar for  $g[=: f](\mathbf{u}, \mathbf{t})$ . The following modification and definition formalize this convention.

**Modification 2.** For certain pairs  $g, f$ , where  $g$  is an external function symbol and  $f$  a reply-available function symbol, an external function symbol  $g[=: f]$  is designated, with arity equal to the sum of the arities of  $g$  and  $f$ .

**Definition 54.** When  $g[=: f]$  is defined, we say that  $f$  is *reply-available for  $g$* . In this case, if  $g$  is  $m$ -ary and  $f$  is  $n$ -ary, then  $g(u_1, \dots, u_m)[=: f(t_1, \dots, t_n)]$  means  $g[=: f](u_1, \dots, u_m, t_1, \dots, t_n)$ .

At this stage, we have ensured that ASM programs written with the  $g(\mathbf{u})[=: f(\mathbf{t})]$  notation are syntactically correct, provided  $f$  is reply-available for  $g$ . As a first step toward semantic correctness, we want them to issue the right queries.

**Intention 55.** When  $g[=: f]$  is defined, the associated template is

$$\widehat{g[=: f]} = \langle \hat{g}, \mathbf{r1}, f, \#(m+1), \dots, \#(m+n) \rangle,$$

where  $g$  is  $m$ -ary and  $f$  is  $n$ -ary.

*Remark 56.* We have, once again, taken some liberties with the bracketing. Without liberties, we would have  $\hat{g} \frown \langle \mathbf{r1}, f, \#(m+1), \dots, \#(m+n) \rangle$ , where  $\frown$  denotes concatenation of sequences. It may also be worth noting that  $\hat{g}$  is the same as  $\hat{g}[\#1, \dots, \#m]$ .

The next modification says that, when the algorithm issues a query that contains the  $\mathbf{r1}$  label, it should get an answer to the “query part” preceding  $\mathbf{r1}$ , since the rest merely specifies a reply location. It turns out that the only change needed in the definitions and postulates from [3] is that, when  $\langle q, \mathbf{r1}, l \rangle$  is caused by an initial segment of a history  $\xi$ , it is not this query itself but rather the initial segment  $q$  that counts as issued.

**Modification 3.** The definition of  $\text{Issued}_X(\xi)$  in Definition 18 is amended as follows.  $\text{Issued}_X(\xi)$  consists of those queries  $q$  such that

- $q$  does not contain  $\mathbf{r1}$ , and
- for some initial segment  $\eta$  of  $\xi$ , either  $\eta \vdash_X q$  or  $\eta \vdash_X \langle q, \mathbf{r1}, l \rangle$  for some sequence  $l$ .

**Intention 57.** The only external function symbols whose templates contain  $\mathbf{r1}$  are those of the form  $g[=: f]$ .

The preceding “Intentions” imply that no template contains more than one occurrence of  $\mathbf{r1}$ . Nevertheless, our modification of the definition of  $\text{Issued}$  can handle queries with several  $\mathbf{r1}$ ’s; the first occurrence of  $\mathbf{r1}$  is the one that counts.

*Remark 58.* It is possible for an ASM program to prescribe two different reply locations for what turns out to be the same query. For example, we might have both  $g(\mathbf{u})[=: f(\mathbf{t})]$  and  $g(\mathbf{u}')[=: f'(\mathbf{t}')] ]$  where, in some (or even every) state  $\mathbf{u}$  and  $\mathbf{u}'$  have the same values  $\mathbf{a}$  but  $f \neq f'$ . Then the queries resulting from these two occurrences are different, but they differ only after the **rl**. So our redefinition of Issued says that only a single query is issued, namely  $\hat{g}[\mathbf{a}]$ . According to Intention 59 below, a reply to this single query is to be written into both of the reply locations.

To see that this is as it should be, consider the ASM program (in the traditional sense) that results from deleting all the reply locations. There,  $g(\mathbf{u})$  and  $g(\mathbf{u}')$  would issue only a single query,  $\hat{g}[\mathbf{a}]$ . Our modifications and definitions ensure that the ASM with reply locations behaves, in this respect, the same as the one without reply locations.

It remains only to formally state, as intentions, the constraints that the environment should obey in order to make our ASMs with persistent queries behave as intended.

**Intention 59.** If the algorithm has produced the query  $\langle q, \mathbf{rl}, l \rangle$ , thereby issuing  $q$ , and if  $l$  is a location, then the answer to  $q$  should be written into location  $l$ . In case several answers are to be put into the same  $l$  at the same time, the environment chooses one of them arbitrarily. The environment should not write into reply locations except as prescribed here.

Usually, a program will not use the same reply location for several different queries, and so the need for an arbitrary choice will not arise. If, however, the program does assign the same reply location to several queries, then not only might it encounter the nondeterminism described here, but replies might be overwritten.

Note that, in telling the algorithm to write replies into the prescribed locations, we have made no exception for on-time replies. If a query is answered during the same step in which it was issued, then the reply goes into both the history of that step and (when the step ends) the reply location.

**Intention 60.** Replies to persistent queries are different from **undef**.

The point of this is to enable an algorithm to detect whether a query has received a late reply. If the value of the reply location is initialized to **undef** and is not updated otherwise than by a reply to  $q$ , then the presence of a reply can be detected by comparing the value of this location to **undef**.

*Remark 61.* We briefly indicate an alternative approach that does not require the environment to reinterpret  $\langle q, \mathbf{rl}, l \rangle$  as the query  $q$  (and does not require us to redefine Issued). In this approach,  $g(\mathbf{u})[=: f(\mathbf{t})]$  should produce two queries, namely the query  $q$  that would be issued by  $g(\mathbf{u})$  alone and the additional query  $\langle q, \mathbf{rl}, l \rangle$  giving its reply location  $l$ . The environment treats  $q$  like any other query, answering it (if possible) in the usual way. It treats  $\langle q, \mathbf{rl}, l \rangle$  as a message, answering it with an automatic, immediate “OK,” but remembering it so that it knows where to write a reply to  $q$  later.

This approach requires a modification to [3] to allow two queries to be caused at a single point in an ASM program. The template assignment should now be multivalued, assigning to  $g[=: f]$  both the template  $\langle \hat{g}, \mathbf{rl}, f, \#(m+1), \dots, \#(m+n) \rangle$  used above and the template  $\hat{g}$ . A secondary modification is to allow an  $m$ -ary template  $\hat{g}$  to be used for an  $(m+n)$ -ary function symbol  $g[=: f]$ .

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